Hawking temperature from tunnelling formalism

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It has recently been suggested that the attempt to understand Hawking radiation as tunnelling across black hole horizons produces a Hawking temperature double the standard value. It is explained here how one can obtain the standard value in the same tunnelling approach.

A classical black hole has a horizon beyond which nothing can leak out. But there is a relation between the area of the horizon and the mass (and other parameters like the charge) indicating a close similarity [1] with the thermodynamical laws, thus allowing the definition of an entropy and a temperature [2]. This analogy was surmised to be of quantum origin and made quantitative after the theoretical discovery of radiation from black holes [3]. For a Schwarzschild black hole, the radiation, which is thermal, has a temperature

$$T_H = \frac{\hbar}{4\pi r_h} = \frac{\hbar}{8\pi M}, \tag{1}$$

where r_h gives the location of the horizon in standard coordinates and M is the mass of the black hole. This was derived by considering quantum massless particles in a Schwarzschild background geometry. The derivation being quite complicated, attempts have been made to understand the process of radiation by other methods. In [4], a path integral study was made, and analytic continuation in complex time used to relate amplitudes for particle emission and absorption with the result that the ratio of emission and absorption probabilities for energy E is given by

$$P_{emission} = \exp(-\frac{E}{T_H})P_{absorption}. \tag{2}$$

This provides further evidence for the temperature T_H. Furthermore, the propagator in the Schwarzschild background was shown [4] to have a periodicity in the imaginary part of time with period $4\pi r_h = 8\pi M$, again suggesting the same temperature. There is also an argument involving a conical singularity on passing to imaginary time, which can only be avoided if the standard Hawking temperature is chosen.

Later, other attempts were made to understand the emission of particles across the horizon as a quantum mechanical tunnelling process [5]. The approach of using (2) was followed in [6]. Different Hamilton - Jacobi treatments were used to reproduce the standard temperature T_H [7]. Recently, however, it has been pointed out [8] that this approach seems to produce a temperature that is *double* the standard value T_H, which corresponds to a halving of the period in imaginary time. This is reminiscent of [9], where it was pointed out that the Hawking temperature could be doubled with a different interpretation of the gravitational field in quantum theory. However, such an interpretation is not used in [8]. So it becomes necessary to try to resolve the contradiction between this and the earlier analyses.

A massless particle in the Schwarzschild background is described by the Klein-Gordon equation

$$\hbar^2(-g)^{-1/2}\partial_\mu(g^{\mu\nu}(-g)^{1/2}\partial_\nu\phi) = 0. \tag{3}$$

One expands

$$\phi = \exp(-\frac{i}{\hbar}S + ...) \tag{4}$$

to obtain to leading order in $\hbar$ the equation

$$g^{\mu\nu}\partial_\mu\partial_\nu S = 0. \tag{5}$$

If we write, for the time being,

$$S = Et + S_0(r), \tag{6}$$

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the equation becomes

$$-\frac{E^2}{1-\frac{r_h}{r}}+(1-\frac{r_h}{r})S_0'(r)^2=0 \tag{7}$$

in the Schwarzschild metric. The formal solution of this equation is

$$S_0(r)=\pm E\int^r\frac{dr}{1-\frac{r_h}{r}}. \tag{8}$$

The sign ambiguity comes from the square root and corresponds to the fact that there can be incoming/outgoing solutions. There is, furthermore, a singularity at the horizon $r=r_h$, which has to be handled if one tries to find a solution across it.

One way to skirt the pole is to change $r-r_h$ to $r-r_h-i\epsilon$. This yields

$$S_0(r)=\pm E[r+r_h\cdot i\pi+r_h\int^r drP(\frac{1}{r-r_h})], \tag{9}$$

where $P()$ denotes the principal value. For the outgoing solution,

$$S_{out}=Et-E[r+r_h\cdot i\pi+r_h\int^r drP(\frac{1}{r-r_h})], \tag{10}$$

the imaginary part yields a decay factor $\exp(-\pi r_h E/\hbar)$ in the amplitude and hence a factor $\exp(-2\pi r_h E/\hbar)$ in the probability. This has been interpreted to signal a temperature [8]

$$\frac{\hbar}{2\pi r_h}=2T_H, \tag{11}$$

twice as big as the standard Hawking temperature.

This observation given in [8] may suggest that one should dump the original calculation [3]. However that calculation has not been directly challenged, nor can one forget the other arguments in support of the standard value of T_H, for example the one involving the periodicity in imaginary time or the conical singularity in passing to imaginary time. So we have to see if it is possible to make sense of the imaginary part of the above S_0 without doubling the Hawking temperature.

A point made in [7] is that r is not the proper radial distance, and ought to be replaced by $\sigma\approx 2\sqrt{r_h(r-r_h)}$ before introducing an $i\epsilon$. However, the use of this variable involves a different kind of path and the evaluation of the integral by [7] has been criticized in [8].

It is more interesting to compare the above argument with [6], where, following [4], (2) is used. Instead of just looking at the outgoing solution, they consider the incoming solution as well:

$$S_{in}=Et+E[r+r_h\cdot i\pi+r_h\int^r drP(\frac{1}{r-r_h})]. \tag{12}$$

The imaginary part here yields a factor $\exp(\pi r_h E/\hbar)$ in the amplitude, leading to a factor $\exp(2\pi r_h E/\hbar)$ in the probability. The *ratio of the outgoing and incoming probabilities* is $\exp(-4\pi r_h E/\hbar)$, which is as in (2). This is how one can think of obtaining the standard temperature instead of getting twice the value. But the curious fact is that the above incoming factor is an *amplification*, not a decay, so that the absorption probability tends to be greater than unity and goes to infinity in the classical limit.

Let us instead rewrite the outgoing and incoming solutions as

$$\begin{aligned}S_{out} &= Et+C-E[r+r_h\cdot i\pi+r_h\int^r drP(\frac{1}{r-r_h})],\\ S_{in} &= Et+C+E[r+r_h\cdot i\pi+r_h\int^r drP(\frac{1}{r-r_h})],\end{aligned} \tag{13}$$

where C is a constant arising from the integration of $\frac{\partial S}{\partial t}$. The real part of C is quite arbitrary, but the imaginary part has to be determined so as to cancel the imaginary part of S_{in}. This is essential to ensure that the incoming probability is unity in the classical limit – when everything is absorbed – instead of zero or infinity. Thus,

$$\begin{aligned}C &= -i\pi r_h E+(Re\ C),\\ S_{out} &= Et-E[r+r_h\cdot 2i\pi+r_h\int^r drP(\frac{1}{r-r_h})]+(Re\ C),\end{aligned} \tag{14}$$

implying a decay factor $\exp(-2\pi r_h E/\hbar)$ in the amplitude, and a factor $\exp(-4\pi r_h E/\hbar)$ in the probability, in conformity with the standard value of the Hawking temperature.

The above calculation has been done in Schwarzschild coordinates. Alternative coordinates that have often been used for tunnelling studies are the ones due to Painlevé [5, 8]. In this case the calculation of S [8] shows that S_{in} has no imaginary part to begin with, so that there is no need to introduce $Im\ C$. One finds further

$$Im\ S_{out} = -2\pi r_h E \tag{15}$$

directly, yielding the expected decay factor $\exp(-4\pi r_h E/\hbar)$ in the probability. There is no lack of consistency [8, 10] between the Schwarzschild and the Painlevé formulations, and it is reassuring to note that $Im\ S_{out} - Im\ S_{in} = -2\pi r_h E$ in both the Schwarzschild and the Painlevé cases, irrespective of the value of the complex constant C.

In short, there is no problem with the standard value of the Hawking temperature. Hawking radiation at the standard temperature can be understood through tunnelling, contrary to the view of [8]. The crucial step is to note that the classical absorption probability is unity.

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